

DELOCALIZED L^2 -INVARIANTS

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ABSTRACT. We define extensions of the L^2 -analytic invariants of closed manifolds, called delocalized L^2 -invariants. These delocalized invariants are constructed in terms of a non-trivial conjugacy class of the fundamental group. We show that in many cases, they are topological in nature. We show that the marked length spectrum of an odd-dimensional hyperbolic manifold can be recovered from its delocalized L^2 -analytic torsion. There are technical convergence questions.

1. INTRODUCTION

Let M^d be a closed d -dimensional hyperbolic manifold. Within the class of such manifolds, the volume $\text{vol}(M)$ of M is a topological invariant; this follows from Mostow rigidity [26] if $d > 2$ and from the Gauss-Bonnet theorem if $d = 2$.

One can ask whether there is a topological invariant defined on a wider class of smooth manifolds which, in the case of a hyperbolic manifold, reproduces its hyperbolic volume. One such invariant is Gromov's simplicial volume [11], which is defined for all compact manifolds M .

If d is even then another relevant topological invariant is the Euler characteristic $\chi(M)$. If M is hyperbolic then $\text{vol}(M) = (-1)^{d/2} \frac{\text{vol}(S^d)}{2} \chi(M)$. Equivalently, we could phrase this in terms of the L^2 -Euler characteristic $\chi_{(2)}(M)$, which equals $\chi(M)$ [1].

An odd-dimensional counterpart is the L^2 -analytic torsion $T_{\langle e \rangle}(M)$ [15, 20], which is a (smooth) topological invariant of compact manifolds M with vanishing L^2 -Betti numbers and positive Novikov-Shubin invariants. Odd-dimensional closed hyperbolic manifolds satisfy these conditions. For such manifolds, $T_{\langle e \rangle}(M) = c_d \text{vol}(M)$ where c_d only depends on d . For example, $c_3 = -\frac{1}{3\pi}$.

Besides the volume, another interesting invariant of a closed hyperbolic manifold is its marked length spectrum. This is the function which, to a conjugacy class $\langle g \rangle \subset \pi_1(M)$, assigns the hyperbolic length of the unique closed geodesic in the free homotopy class specified by $\langle g \rangle$. In this paper we address the question of whether there is a topological invariant of some class of smooth manifolds which, in the case of a hyperbolic manifold of dimension greater than two, reproduces its marked length spectrum.

We consider three invariants of a closed Riemannian manifold M , called the delocalized L^2 -Betti numbers, L^2 -analytic torsion and L^2 -eta invariant. These invariants are defined in Section 2. The reason for the word "delocalized" is that the ordinary L^2 -invariants [1, 5, 15, 20] are localized at the identity element of the fundamental group, in a sense which will be made precise. In contrast, the delocalized invariants are defined in terms of a nontrivial conjugacy class $\langle g \rangle$ of the fundamental group.

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The delocalized L^2 -invariants are constructed using the heat kernel on the universal cover \widetilde{M} of M . There are technical difficulties in showing that the formal expressions for the invariants are actually well-defined. These difficulties involve estimating heat kernels on \widetilde{M} at large-distance and large-time simultaneously. We cannot show that the formal expressions make sense in full generality. However, we do not know of any examples in which this is not the case. We do show that the delocalized L^2 -invariants are well-defined and metric-independent in the following cases :

1. If $\langle g \rangle$ is a finite conjugacy class.
2. If the fundamental group is virtually nilpotent or Gromov-hyperbolic and if there is a gap away from zero in the spectrum of the lifted p -form Laplacian, in the case of the delocalized L^2 -Betti number.
3. If the fundamental group is virtually nilpotent or Gromov-hyperbolic and if there is a gap away from zero in the spectrum of the lifted Dirac-type operator, in the case of the delocalized L^2 -eta invariant.

We compute the delocalized L^2 -invariants in some cases of interest. For an odd-dimensional hyperbolic manifold, we show that the delocalized L^2 -invariants are well-defined and that the delocalized L^2 -analytic torsion reproduces the marked length spectrum. This is based on work of Fried [8] and Millson [21]. In the case of a mapping torus with simply-connected fiber, we show that the delocalized L^2 -invariants can be written in terms of the action of the gluing diffeomorphism on the fiber cohomology. In the case of a mapping torus whose fiber has finite fundamental group, we show that the delocalized L^2 -analytic torsion is determined by the Nielsen fixed point indices of the gluing diffeomorphism.

If the fundamental group is torsion-free then the delocalized L^2 -Betti numbers vanish in all cases in which we can compute them. We do not know if this is always the case.

In order to really say that we have defined topological invariants which reproduce the marked length spectrum of an odd-dimensional hyperbolic manifold, it remains to prove the following.

Conjecture 1. *Let M be a closed odd-dimensional manifold which admits a hyperbolic structure. Then for any Riemannian metric on M , the delocalized L^2 -invariants are well-defined and independent of the choice of metric.*

Another interesting question is whether there is a “delocalized” version of the simplicial volume.

The structure of the paper is as follows. In Section 2 we give the definitions of the invariants and the statements of the main results. In Sections 3-7 we prove the results. In Section 8 we give examples to show that the results are not vacuous.

2. DEFINITIONS AND STATEMENTS OF RESULTS

Let M^d be a closed connected oriented Riemannian manifold. Let $\pi : \widehat{M} \rightarrow M$ be a connected normal Γ -cover of M , equipped with the pullback Riemannian metric. We let $\gamma \in \Gamma$ act on \widehat{M} on the right by $R_\gamma \in \text{Diff}(\widehat{M})$. Let \mathcal{C} denote the set of conjugacy classes of Γ .

Definition 1. *Let \mathcal{A} be the convolution algebra of elements $a \in C^\infty(\widehat{M} \times \widehat{M})$ satisfying*

1. $a(m\gamma, m'\gamma) = a(m, m')$ for all $\gamma \in \Gamma$ and

2. There exists an $R_a > 0$ such that if $d(m, m') \geq R_a$ then $a(m, m') = 0$.

The multiplication in \mathcal{A} is given by

$$(ab)(m, m') = \int_{\widehat{M}} a(m, m'') b(m'', m') d\text{vol}(m''). \quad (2.1)$$

Given $a \in \mathcal{A}$ and $\langle g \rangle \in \mathcal{C}$, define $A_{\langle g \rangle} \in C^\infty(\widehat{M})$ by

$$A_{\langle g \rangle}(m) = \sum_{\gamma \in \langle g \rangle} a(m\gamma, m). \quad (2.2)$$

Lemma 1. For all $\gamma' \in \Gamma$,

$$A_{\langle g \rangle}(m\gamma') = A_{\langle g \rangle}(m). \quad (2.3)$$

Proof. We have

$$A_{\langle g \rangle}(m\gamma') = \sum_{\gamma \in \langle g \rangle} a(m\gamma'\gamma, m\gamma') = \sum_{\gamma \in \langle g \rangle} a\left(m\gamma'\gamma(\gamma')^{-1}, m\right) = A_{\langle g \rangle}(m). \quad (2.4)$$

□

Thus $A_{\langle g \rangle} = \pi^* \alpha_{\langle g \rangle}$ for a unique $\alpha_{\langle g \rangle} \in C^\infty(M)$.

Definition 2. Define $\text{Tr}_{\langle g \rangle} : \mathcal{A} \rightarrow \mathbb{C}$ by

$$\text{Tr}_{\langle g \rangle}(a) = \int_M \alpha_{\langle g \rangle} d\text{vol}. \quad (2.5)$$

Lemma 2. For all $a, b \in \mathcal{A}$,

$$\text{Tr}_{\langle g \rangle}(ab) = \text{Tr}_{\langle g \rangle}(ba). \quad (2.6)$$

Proof. Let \mathcal{F} be a fundamental domain for the Γ -action on \widehat{M} . Formally,

$$\begin{aligned}
\mathrm{Tr}_{\langle g \rangle}(ab) &= \int_{\mathcal{F}} \sum_{\gamma \in \langle g \rangle} (ab)(m\gamma, m) \, d\mathrm{vol}(m) \\
&= \int_{\mathcal{F}} \sum_{\gamma \in \langle g \rangle} \int_{\widehat{M}} a(m\gamma, m') b(m', m) \, d\mathrm{vol}(m') \, d\mathrm{vol}(m) \\
&= \int_{\mathcal{F}} \sum_{\gamma \in \langle g \rangle} \int_{\mathcal{F}} \sum_{\gamma' \in \Gamma} a(m\gamma, m'\gamma') b(m'\gamma', m) \, d\mathrm{vol}(m') \, d\mathrm{vol}(m) \\
&= \sum_{\gamma \in \langle g \rangle} \sum_{\gamma' \in \Gamma} \int_{\mathcal{F}} \int_{\mathcal{F}} b(m'\gamma', m) a(m\gamma, m'\gamma') \, d\mathrm{vol}(m') \, d\mathrm{vol}(m) \\
&= \sum_{\gamma \in \langle g \rangle} \sum_{\gamma' \in \Gamma} \int_{\mathcal{F}} \int_{\mathcal{F}} b(m'\gamma'\gamma(\gamma')^{-1}, m\gamma(\gamma')^{-1}) a(m\gamma(\gamma')^{-1}, m') \, d\mathrm{vol}(m') \, d\mathrm{vol}(m) \\
&= \sum_{\gamma \in \langle g \rangle} \sum_{\gamma'' \in \Gamma} \int_{\mathcal{F}} \int_{\mathcal{F}} b(m'(\gamma'')^{-1}\gamma\gamma'', m\gamma'') a(m\gamma'', m') \, d\mathrm{vol}(m') \, d\mathrm{vol}(m) \\
&= \sum_{\gamma \in \langle g \rangle} \sum_{\gamma'' \in \Gamma} \int_{\mathcal{F}} \int_{\mathcal{F}} b(m'\gamma, m\gamma'') a(m\gamma'', m') \, d\mathrm{vol}(m) \, d\mathrm{vol}(m') \\
&= \sum_{\gamma \in \langle g \rangle} \int_{\mathcal{F}} \int_{\widehat{M}} b(m'\gamma, m) a(m, m') \, d\mathrm{vol}(m) \, d\mathrm{vol}(m') \\
&= \mathrm{Tr}_{\langle g \rangle}(ba).
\end{aligned} \tag{2.7}$$

It is not hard to justify the steps in (2.7). \square

We will need two slight extensions of the algebra \mathcal{A} . First, let E be a Hermitian vector bundle on M . Put $\widehat{E} = \pi^*E$. For $i \in \{1, 2\}$, let π_i be projection from $\widehat{M} \times \widehat{M}$ onto the i -th factor of \widehat{M} . Let \mathcal{A} be the convolution algebra of elements $a \in C^\infty(\widehat{M} \times \widehat{M}; \pi_1^*\widehat{E} \otimes \pi_2^*\widehat{E}^*)$ satisfying the two conditions of Definition 1. Equation (2.2) now has to be interpreted as

$$A_{\langle g \rangle}(m) = \sum_{\gamma \in \langle g \rangle} \mathrm{tr}((R_\gamma^*a)(m, m)) = \sum_{\gamma \in \langle g \rangle} \mathrm{tr}(a(m\gamma, m)). \tag{2.8}$$

Then the proof of Lemma 2 extends. Next, we can replace condition 2. of Definition 1 by the weaker assumption that

$$\text{for all } R > 0, \quad \sup_{d(m, m') \leq R} \sum_{\gamma} |a(m\gamma, m')| < \infty. \tag{2.9}$$

Equation (2.9) is essentially an l^1 -condition on a with respect to Γ . Then we again have a convolution algebra and the proof of Lemma 2 still goes through.

Let $\widehat{\Delta}_p$ be the p -form Laplacian on \widehat{M} . For $t > 0$, let $e^{-t\widehat{\Delta}_p}$ be the corresponding heat operator. It has a Schwartz kernel $e^{-t\widehat{\Delta}_p}(m, m') \in \Lambda^p(T_m^*\widehat{M}) \otimes (\Lambda^p(T_{m'}^*\widehat{M}))^*$. By finite propagation speed estimates [6], $e^{-t\widehat{\Delta}_p}(m, m')$ satisfies (2.9).

Definition 3. Take $g \neq e$. When the limit exists, we define the p -th delocalized L^2 -Betti number of M by

$$b_{p,\langle g \rangle}(M) = \lim_{t \rightarrow \infty} \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} \right) \quad (2.10)$$

If we were to take $g = e$ then $b_{p,\langle e \rangle}(M)$ would be the same as the p -th L^2 -Betti number of M [1].

Let $\{ds^2(u)\}_{u \in [-1,1]}$ be a smooth 1-parameter family of Riemannian metrics on M . Let $\{*(u)\}_{u \in [-1,1]}$ be the corresponding 1-parameter family of Hodge duality operators. Then

$$\begin{aligned} \frac{d}{du} \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} \right) &= -t \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} \frac{d\hat{\Delta}_p}{du} \right) \\ &= -t \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} \left(d \left[d^*, *^{-1} \frac{d*}{du} \right] + \left[d^*, *^{-1} \frac{d*}{du} \right] d \right) \right) \\ &= -2t \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} (dd^* - d^*d) *^{-1} \frac{d*}{du} \right). \end{aligned} \quad (2.11)$$

Let $\Pi_{\text{Ker}(\hat{\Delta}_p)}$ be projection onto the L^2 -kernel of $\hat{\Delta}_p$. As $\lim_{t \rightarrow \infty} e^{-t\hat{\Delta}_p} = \Pi_{\text{Ker}(\hat{\Delta}_p)}$ and

$$\Pi_{\text{Ker}(\hat{\Delta}_p)} dd^* = \Pi_{\text{Ker}(\hat{\Delta}_p)} d^*d = 0, \quad (2.12)$$

we expect that

$$\lim_{t \rightarrow \infty} -2t \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} (dd^* - d^*d) *^{-1} \frac{d*}{du} \right) = 0. \quad (2.13)$$

In summary, we have shown the following result.

Proposition 1. If (2.13) can be justified, uniformly in u , then $b_{p,\langle g \rangle}(M)$ is metric-independent and hence a (smooth) topological invariant of M .

For the moment, let us assume that $b_{p,\langle g \rangle}(M)$ is metric-independent. For $t > 0$, put

$$\mathcal{T}_{\langle g \rangle}(t) = \sum_{p=0}^d (-1)^p p \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} \right). \quad (2.14)$$

Put

$$\mathcal{T}_{\langle g \rangle}(\infty) = \sum_{p=0}^d (-1)^p p b_{p,\langle g \rangle}(M). \quad (2.15)$$

Definition 4. Take $g \neq e$. When the integral makes sense, we define the delocalized L^2 -analytic torsion by

$$\mathcal{T}_{\langle g \rangle}(M) = - \int_0^\infty \left(\mathcal{T}_{\langle g \rangle}(t) - (1 - e^{-t}) \mathcal{T}_{\langle g \rangle}(\infty) \right) \frac{dt}{t}. \quad (2.16)$$

If we were to take $g = e$ then $\mathcal{T}_{\langle g \rangle}(M)$ would formally be the same, up to a sign, as the L^2 -analytic torsion of [15, 20]. (The latter requires a zeta-function regularization in its definition, but this is not necessary for the delocalized L^2 -analytic torsion.) It follows from

finite propagation speed arguments that the integrand in (2.16) is integrable for small t . Thus the question of whether the integral makes sense refers to large- t integrability.

Let $\{ds^2(u)\}_{u \in [-1,1]}$ be a smooth 1-parameter family of Riemannian metrics on M . Then for any $t > 0$, as in the proof of [15, Lemma 8], we have

$$\frac{d}{du} \mathcal{T}_{\langle g \rangle}(t) = t \frac{d}{dt} \sum_{p=0}^d (-1)^p \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} *^{-1} \frac{d*}{du} \right). \quad (2.17)$$

Proceeding formally,

$$\frac{d}{du} \mathcal{T}_{\langle g \rangle}(M) = \left(\lim_{t \rightarrow 0} - \lim_{t \rightarrow \infty} \right) \sum_{p=0}^d (-1)^p \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} *^{-1} \frac{d*}{du} \right). \quad (2.18)$$

As we are assuming that $g \neq e$, it follows again from finite propagation speed estimates that

$$\lim_{t \rightarrow 0} \sum_{p=0}^d (-1)^p \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} *^{-1} \frac{d*}{du} \right) = 0. \quad (2.19)$$

When it can be justified, we expect that

$$\lim_{t \rightarrow \infty} \sum_{p=0}^d (-1)^p \text{Tr}_{\langle g \rangle} \left(e^{-t\hat{\Delta}_p} *^{-1} \frac{d*}{du} \right) = \sum_{p=0}^d (-1)^p \text{Tr}_{\langle g \rangle} \left(\Pi_{\text{Ker}(\hat{\Delta}_p)} *^{-1} \frac{d*}{du} \right). \quad (2.20)$$

Now $\text{Ker}(\hat{\Delta}_p)$ can be identified with the p -dimensional (reduced) L^2 -cohomology group of M and is topological in nature [1]. In summary, we have shown the following result.

Proposition 2. *If (2.20) can be justified, uniformly in u , and if M has vanishing L^2 -cohomology then for all $g \neq e$, $\mathcal{T}_{\langle g \rangle}(M)$ is a (smooth) topological invariant of M .*

Now let \hat{E} be a Γ -invariant Clifford module on \hat{M} . For simplicity, we assume that M is spin, with S denoting the spinor bundle, and that there is a Hermitian vector bundle V on M so that $\hat{E} = \pi^*(S \otimes V)$; the general case is similar. Let ∇^S be the connection on S coming from the Levi-Civita connection and let ∇^V be a Hermitian connection on V . Let D be the corresponding self-adjoint Dirac-type operator on sections of $S \otimes V$ and let \hat{D} be the lifted operator on sections of \hat{E} . For $t > 0$, let $e^{-t\hat{D}^2}$ be the corresponding heat operator. It has a Schwartz kernel $e^{-t\hat{D}^2}(m, m') \in \hat{E}_m \otimes \hat{E}_{m'}^*$. Again, using finite propagation estimates it is not hard to see that $e^{-t\hat{D}^2}$ satisfies (2.9). Given a conjugacy class $\langle g \rangle$ in Γ , for $s > 0$ put

$$\eta_{\langle g \rangle}(s) = \text{Tr}_{\langle g \rangle} \left(\hat{D} e^{-s^2 \hat{D}^2} \right). \quad (2.21)$$

Definition 5. *Take $g \neq e$. When the integral makes sense, we define the delocalized L^2 -eta invariant by*

$$\eta_{\langle g \rangle}(M) = \frac{2}{\sqrt{\pi}} \int_0^\infty \eta_{\langle g \rangle}(s) ds. \quad (2.22)$$

If we were to take $g = e$ then $\eta_{\langle g \rangle}(M)$ would be the same as the L^2 -eta invariant of Cheeger and Gromov [5]. In this case, it is known that the integral in (2.22) makes sense [3, 29, 30]. If $g \neq e$ then finite propagation speed arguments show that the integrand in (2.22) is integrable for small- s . Thus the question of whether the integral makes sense refers to large- s integrability. Equation (2.22) was first considered, when Γ is virtually nilpotent, in [14, Eqn. (69)].

Let $\{ds^2(u)\}_{u \in [-1,1]}$, $\{h^V(u)\}_{u \in [-1,1]}$ and $\{\nabla^V(u)\}_{u \in [-1,1]}$ be smooth 1-parameter families of Riemannian metrics on M , Hermitian metrics on V and compatible Hermitian connections on V , respectively. Then for any $s > 0$, one can check that

$$\frac{d}{du}\eta_{\langle g \rangle}(s) = \frac{d}{ds}\mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right). \quad (2.23)$$

Proceeding formally,

$$\frac{d}{du}\eta_{\langle g \rangle}(M) = \frac{2}{\sqrt{\pi}} \left(\lim_{s \rightarrow \infty} - \lim_{s \rightarrow 0} \right) \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right). \quad (2.24)$$

As we are assuming that $g \neq e$, it follows from finite propagation speed estimates that

$$\frac{2}{\sqrt{\pi}} \lim_{s \rightarrow 0} \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right) = 0. \quad (2.25)$$

In summary, we have shown the following result.

Proposition 3. *If*

$$\frac{2}{\sqrt{\pi}} \lim_{s \rightarrow \infty} \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right) = 0 \quad (2.26)$$

uniformly in u then $\eta_{\langle g \rangle}(M)$ is independent of u .

Proceeding very formally,

$$\frac{2}{\sqrt{\pi}} \lim_{s \rightarrow \infty} \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right) = 2\mathrm{Tr}_{\langle g \rangle} \left(\frac{d\hat{D}}{du} \delta(\hat{D}) \right) = 2 \frac{d}{du} \mathrm{Tr}_{\langle g \rangle} \left(\mathrm{sign}(\hat{D}) \right). \quad (2.27)$$

Thus we expect that if $\mathrm{sign}(\hat{D})$ is independent of u , as a Γ -operator, then $\eta_{\langle g \rangle}(M)$ is independent of u . In particular, we expect that this will be true in the following cases :

1. If D is the Dirac operator and for all $u \in [-1, 1]$, $(M, ds^2(u))$ has positive scalar curvature.
2. If D is the (tangential) signature operator of [2].

We now give some elementary properties of the delocalized L^2 -invariants.

Proposition 4. *Suppose that Γ is finite. Let $\{\langle g_i \rangle\}$ parametrize the conjugacy classes of Γ . Let $\rho : \Gamma \rightarrow U(N)$ be a unitary representation of Γ . Let E_ρ be the associated flat Hermitian vector bundle on M . Let $\chi_\rho : \Gamma \rightarrow \mathbb{C}$ be the character of ρ .*

1. *Then*

$$b_p(M; E_\rho) = \sum_i \chi_\rho(g_i) b_{p, \langle g_i \rangle}(M). \quad (2.28)$$

2. Let $\mathcal{T}(M; E_\rho) \in \mathbb{R}$ be the Ray-Singer analytic torsion [31]. Then

$$\mathcal{T}(M; E_\rho) = \sum_i \chi_\rho(g_i) \mathcal{T}_{\langle g_i \rangle}(M). \quad (2.29)$$

3. Let D be a Dirac-type operator on M . Let $\eta(M; E_\rho) \in \mathbb{R}$ be the Atiyah-Patodi-Singer eta-invariant [2]. Then

$$\eta(M; E_\rho) = \sum_i \chi_\rho(g_i) \eta_{\langle g_i \rangle}(M). \quad (2.30)$$

Proof. This follows from Fourier analysis on Γ , as in [18, Section 2]. We omit the details. \square

Proposition 4.3 shows that when Γ is a finite group, the delocalized L^2 -eta invariant has the same information as the ρ -invariant of [2].

Proposition 5. 1. We have $b_{p, \langle g^{-1} \rangle}(M) = \overline{b_{p, \langle g \rangle}(M)}$, $\mathcal{T}_{\langle g^{-1} \rangle}(M) = \overline{\mathcal{T}_{\langle g \rangle}(M)}$ and $\eta_{\langle g^{-1} \rangle}(M) = \overline{\eta_{\langle g \rangle}(M)}$.

2. Given pairs (M_1, Γ_1) and (M_2, Γ_2) , we have

$$\mathcal{T}_{\langle g_1, g_2 \rangle}(M_1 \times M_2) = \delta_{g_1, e_1} \chi(M_1) \mathcal{T}_{\langle g_2 \rangle}(M_2) + \delta_{g_2, e_2} \chi(M_2) \mathcal{T}_{\langle g_1 \rangle}(M_1) \quad (2.31)$$

and

$$\begin{aligned} \eta_{\langle g_1, g_2 \rangle}(M_1 \times M_2) = & \delta_{g_1, e_1} \left(\int_{M_1} \widehat{A}(TM_1) \cup \text{ch}(E_1) \right) \eta_{\langle g_2 \rangle}(M_2) + \\ & \delta_{g_2, e_2} \left(\int_{M_2} \widehat{A}(TM_2) \cup \text{ch}(E_2) \right) \eta_{\langle g_1 \rangle}(M_1). \end{aligned} \quad (2.32)$$

3. If d is even then $\mathcal{T}_{\langle g \rangle}(M) = 0$.

4. Suppose that d is odd and D is the (tangential) signature operator [2]. Then $\eta_{\langle g \rangle}(M) = 0$ if $d \equiv 1 \pmod{4}$.

Proof. As $e^{-t\Delta_p}$ is self-adjoint and Γ -invariant,

$$\begin{aligned} \text{tr} \left(e^{-t\widehat{\Delta}_p}(m\gamma^{-1}, m) \right) &= \text{tr} \left(e^{-t\widehat{\Delta}_p}(m, m\gamma) \right) = \text{tr} \left(e^{-t\widehat{\Delta}_p}(m\gamma, m) \right)^* \\ &= \overline{\text{tr} \left(e^{-t\widehat{\Delta}_p}(m\gamma, m) \right)}. \end{aligned} \quad (2.33)$$

It follows that $b_{p, \langle g^{-1} \rangle}(M) = \overline{b_{p, \langle g \rangle}(M)}$. The proof of the rest of 1. is similar. The proofs of 2., 3. and 4. follow from arguments as in [2] and [15]. We omit the details. \square

The main results of this paper are given by the following propositions.

Proposition 6. Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class. Then $b_{p, \langle g \rangle}(M)$ is well-defined and metric-independent. If Γ is a free abelian group then $b_{p, \langle g \rangle}(M) = 0$.

Proposition 7. Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class. Suppose that M has positive Novikov-Shubin invariants [16, 28]. Then the integrand in (2.16) is integrable. If M has vanishing L^2 -cohomology groups then $\mathcal{T}_{\langle g \rangle}(M)$ is metric-independent.

Proposition 8. *Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class and D is a Dirac-type operator. Then the integrand in (2.22) is integrable. If D is the (tangential) signature operator then $\eta_{\langle g \rangle}(M)$ is metric-independent. If D is the Dirac operator then $\eta_{\langle g \rangle}(M)$ is a locally-constant function on the space of positive-scalar-curvature metrics on M .*

Proposition 9. *Suppose that Γ is virtually nilpotent or Gromov-hyperbolic. Suppose that $0 \notin \text{spec}(\hat{\Delta}_p)$ or that 0 is isolated in $\text{spec}(\hat{\Delta}_p)$. Then for all nontrivial conjugacy classes $\langle g \rangle$ of Γ , $b_{p,\langle g \rangle}(M)$ is well-defined and metric-independent.*

Proposition 10. *Suppose that Γ is virtually nilpotent or Gromov-hyperbolic. Suppose that $0 \notin \text{spec}(\hat{D})$ or that 0 is isolated in $\text{spec}(\hat{D})$. Then for all nontrivial conjugacy classes $\langle g \rangle$ of Γ , the integrand in (2.22) is integrable. Furthermore, if D is the Dirac operator then $\eta_{\langle g \rangle}(M)$ is a locally-constant function on the space of positive-scalar-curvature metrics on M .*

Proposition 11. *Let M^d be a closed oriented hyperbolic manifold. Let $\langle g \rangle$ be a nontrivial conjugacy class in $\pi_1(M)$. Then $b_{p,\langle g \rangle}(M) = 0$ for all p .*

Suppose that $d = 2n + 1$. Then $\mathcal{T}_{\langle g \rangle}(M)$ and $\eta_{\langle g \rangle}(M)$ are well-defined, the latter being constructed with the (tangential) signature operator. Let c be the unique closed geodesic in the free homotopy class specified by $\langle g \rangle$. Let $k \in \mathbb{Z}^+$ be the multiplicity of c , meaning the number of times that c covers a prime closed geodesic. Let l be the hyperbolic length of c . Let $m \in SO(2n)$ be the parallel transport of a normal vector around c . Let $\sigma_j(m) \in SO(\Lambda^j(\mathbb{R}^{2n}))$ be the action of m on the exterior power $\Lambda^j(\mathbb{R}^{2n})$. Then

$$\mathcal{T}_{\langle g \rangle}(M) = \frac{e^{-nl}}{k \det(I - e^{-l}m)} \sum_{j=0}^{2n} (-1)^j e^{-l|n-j|} \text{Tr}(\sigma_j(m)). \quad (2.34)$$

In particular, the marked length spectrum of M can be recovered from $\{\mathcal{T}_{\langle g \rangle}(M)\}_{\langle g \rangle \in \mathcal{C}}$.

If n is even then $\eta_{\langle g \rangle}(M) = 0$. If n is odd, define angles $\{\theta_j\}_{j=1}^n$ by saying that the diagonalization of m consists of the 2×2 blocks

$$\begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix}. \quad (2.35)$$

Put $\mu_j = e^{\frac{l+i\theta_j}{2}}$. Then

$$\eta_{\langle g \rangle}(M) = \frac{(2i)^{n+1}}{2\pi k} \frac{\sin(\theta_1) \dots \sin(\theta_n)}{|\mu_1 - \mu_1^{-1}|^2 \dots |\mu_n - \mu_n^{-1}|^2}. \quad (2.36)$$

If $n = 1$ then

$$\mathcal{T}_{\langle g \rangle}(M) - i\pi\eta_{\langle g \rangle}(M) = \frac{2}{k} \frac{1}{1 - \mu_1^2}. \quad (2.37)$$

Proposition 12. *Let Z^n be a smooth closed even-dimensional manifold and let ϕ be a diffeomorphism of Z . Let $\phi_p^* \in \text{Aut}(H^p(Z; \mathbb{C}))$ be the induced map on cohomology. Let M be the mapping torus*

$$M = (Z \times [0, 1]) / \{(z, 0) \sim (\phi(z), 1)\}. \quad (2.38)$$

Put $\Gamma = \mathbb{Z}$, acting on the cyclic cover \widehat{M} of M . Define $f : \mathbb{C} \rightarrow \mathbb{C}$ by

$$f(\lambda) = \begin{cases} \lambda & \text{if } |\lambda| \leq 1, \\ \frac{\lambda}{\lambda^{-1}} & \text{if } |\lambda| > 1. \end{cases} \quad (2.39)$$

Then

$$\mathcal{T}_{\langle k \rangle}(M) = \begin{cases} \frac{1}{k} \sum_{p=0}^d (-1)^p \operatorname{Tr} [f(\phi_p^*)]^k & \text{if } k > 0, \\ -\frac{1}{k} \sum_{p=0}^d (-1)^p \operatorname{Tr} [f(\overline{\phi_p})]^{-k} & \text{if } k < 0. \end{cases} \quad (2.40)$$

Equivalently, let $L(\phi^k)$ be the Lefschetz number of ϕ^k . For $z \in \mathbb{C}$ with $|z|$ small, put $\zeta(z) = \exp \left(\sum_{k>0} \frac{z^k}{k} L(\phi^k) \right)$. Then $\zeta(z)$ has a meromorphic continuation to $z \in \mathbb{C}$, and

$$\mathcal{T}_{\langle k \rangle}(M) = \int_{S^1} e^{-ik\theta} \ln |\zeta(e^{i\theta})|^2 \frac{d\theta}{2\pi}. \quad (2.41)$$

Suppose that ϕ preserves a Dirac-type operator D_Z on Z . Let D be the suspended Dirac-type operator on M . Then $\eta_{\langle k \rangle}(M)$ is given in terms of the Atiyah-Bott indices by

$$\eta_{\langle k \rangle}(M) = \frac{i}{k\pi} \operatorname{Tr}_s \left(\phi^k \Big|_{\operatorname{Ker}(D_Z)} \right). \quad (2.42)$$

Proposition 13. Let Z^n be a smooth closed even-dimensional manifold. Let F be a finite group and let \widehat{Z} be a connected normal F -cover of Z . Let ϕ be a diffeomorphism of Z and let M be the mapping torus of ϕ . Let $\widehat{\phi}$ be a lift of ϕ to \widehat{Z} and let $\alpha \in \operatorname{Aut}(F)$ be the automorphism defined by

$$\widehat{\phi}(zf) = \widehat{\phi}(z)\alpha^{-1}(f) \quad (2.43)$$

for all $z \in \widehat{Z}$ and $f \in F$. Put $\Gamma = F \widetilde{\times}_\alpha \mathbb{Z}$, acting on $\widehat{Z} \times \mathbb{R}$ on the right by

$$(z, t) \cdot (f, k) = (\widehat{\phi}^k(zf), t + k). \quad (2.44)$$

For $k \in \mathbb{Z}$, define an equivalence relation \sim_k on F by saying that $f \sim_k f'$ if there exists a $\gamma \in F$ such that $\gamma f \alpha^k(\gamma^{-1}) = f'$. Let $[f]_k$ be the equivalence class of $f \in F$ and let $|[f]_k|$ be its cardinality. Let $I_k(f) \in \mathbb{Z}$ be the Nielsen fixed point index of ϕ^k , evaluated at $[f]_k$. If ρ is a finite-dimensional irreducible unitary representation of $F \widetilde{\times}_\alpha \mathbb{Z}$, let χ_ρ be its character. For $z \in \mathbb{C}$ with $|z|$ small, put

$$\zeta_\rho(z) = \exp \left(\sum_{\substack{f \in F \\ k > 0}} \frac{z^k}{k} \chi_\rho(f, k) \frac{I_k(f)}{|[f]_k|} \right). \quad (2.45)$$

Then $\zeta_\rho(z)$ has a meromorphic continuation to $z \in \mathbb{C}$ and

$$\sum_{f, k} \chi_\rho(f, k) \mathcal{T}_{\langle f, k \rangle}(M) = \ln |\zeta_\rho(1)|^2. \quad (2.46)$$

Knowing (2.46) for all ρ determines $\{\mathcal{T}_{\langle f, k \rangle}(M)\}_{(f, k) \in \Gamma}$.

Before proceeding with the proofs, let us mention why the existence problem for the delocalized L^2 -invariants is more difficult than for the ordinary L^2 -invariants. The algebraic origin of the problem is as follows. Let Γ be a countable discrete group and consider the group algebra $\mathbb{C}\Gamma$ of finite sums $\sum_{\gamma \in \Gamma} c_\gamma \gamma$, with $c_\gamma \in \mathbb{C}$. Define an involution on $\mathbb{C}\Gamma$ by

$$\left(\sum_{\gamma \in \Gamma} c_\gamma \gamma \right)^* = \sum_{\gamma \in \Gamma} \overline{c_\gamma} \gamma^{-1}. \quad (2.47)$$

If $\langle g \rangle$ is a conjugacy class in Γ , the linear functional $\tau_{\langle g \rangle} : \mathbb{C}\Gamma \rightarrow \mathbb{C}$ given by

$$\tau_{\langle g \rangle} \left(\sum_{\gamma \in \Gamma} c_\gamma \gamma \right) = \sum_{\gamma \in \langle g \rangle} c_\gamma \quad (2.48)$$

satisfies $\tau_{\langle g \rangle}(ab) = \tau_{\langle g \rangle}(ba)$ for all $a, b \in \mathbb{C}\Gamma$. Furthermore, if $g = e$ then $\tau_{\langle e \rangle}(a^*a) \geq 0$ and $\tau_{\langle e \rangle}$ extends to a continuous linear functional on the group von Neumann algebra. These last two properties of $\tau_{\langle e \rangle}$, which are crucial for the usual L^2 -invariants, generally fail for $\tau_{\langle g \rangle}$ if $g \neq e$.

3. PROOFS OF PROPOSITIONS 6, 7 AND 8

Let $\langle g \rangle$ be a finite conjugacy class in Γ . Put

$$A = \sum_{\gamma \in \langle g \rangle} R_\gamma^*. \quad (3.1)$$

Then A is a bounded operator on $\Omega^*(\widehat{M})$ which commutes with $R_{\gamma'}$ for all $\gamma' \in \Gamma$. Letting Tr_Γ denote the II_∞ -trace [1], we have

$$\text{Tr}_{\langle g \rangle} \left(e^{-t\widehat{\Delta}_p} \right) = \text{Tr}_\Gamma \left(A e^{-t\widehat{\Delta}_p} \right). \quad (3.2)$$

Thus

$$b_{p, \langle g \rangle}(M) = \text{Tr}_\Gamma \left(A \Pi_{\text{Ker}(\widehat{\Delta}_p)} \right). \quad (3.3)$$

Hence $b_{p, \langle g \rangle}(M)$ is well-defined. It follows by standard arguments that it is metric-independent.

Similarly,

$$\mathcal{T}_{\langle g \rangle}(t) = \sum_{p=0}^n (-1)^p p \text{Tr}_\Gamma \left(A e^{-t\widehat{\Delta}_p} \right). \quad (3.4)$$

Then the integrand of (2.16) is

$$\frac{1}{t} \sum_{p=0}^n (-1)^p p \text{Tr}_\Gamma \left[A \left(e^{-t\widehat{\Delta}_p} - (1 - e^{-t}) \Pi_{\text{Ker}(\widehat{\Delta}_p)} \right) \right]. \quad (3.5)$$

Now

$$\left| \text{Tr}_\Gamma \left[A \left(e^{-t\widehat{\Delta}_p} - \Pi_{\text{Ker}(\widehat{\Delta}_p)} \right) \right] \right| \leq \|A\| \text{Tr}_\Gamma \left[\left(e^{-t\widehat{\Delta}_p} - \Pi_{\text{Ker}(\widehat{\Delta}_p)} \right) \right]. \quad (3.6)$$

By assumption, there is an $\alpha_p > 0$ such that for large t ,

$$\text{Tr}_\Gamma \left[e^{-t\widehat{\Delta}_p} - \Pi_{\text{Ker}(\widehat{\Delta}_p)} \right] \leq t^{-\alpha_p/2}. \quad (3.7)$$

It follows that the integrand in (2.16) is integrable. If M has vanishing L^2 -cohomology groups then it follows as in [15] that $\mathcal{T}_{\langle g \rangle}(M)$ is metric-independent.

Now let D be a Dirac-type operator. Then we have

$$\eta_{\langle g \rangle}(s) = \text{Tr}_{\Gamma} \left(A \widehat{D} e^{-s^2 \widehat{D}^2} \right) \quad (3.8)$$

Thus

$$\left| \eta_{\langle g \rangle}(s) \right| \leq \|A\| \text{Tr}_{\Gamma} \left(|\widehat{D}| e^{-s^2 \widehat{D}^2} \right). \quad (3.9)$$

It follows as in [30, Theorem 3.1.1] that the integrand in (2.22) is integrable. The rest of Proposition 8 follows as in [5].

Finally, to finish the proof of Proposition 6, suppose that $\Gamma = \mathbb{Z}^l$. Let $\widehat{\Gamma}$ be the Pontryagin dual of Γ . Then an element ρ_{θ} of $\widehat{\Gamma}$ is a representation $\rho_{\theta} : \Gamma \rightarrow U(1)$ of the form

$$\rho_{\theta}(\vec{k}) = e^{i\vec{k} \cdot \vec{\theta}}. \quad (3.10)$$

Let E_{θ} be the associated flat unitary line bundle on M and let $\Delta_{p,\theta}$ be the Laplacian on $\Omega^p(M; E_{\theta})$. It follows as in [15, Proposition 38] that

$$\text{Tr}_{\langle \vec{m} \rangle} \left(e^{-t \widehat{\Delta}_p} \right) = \int_{\widehat{\Gamma}} e^{-i\vec{m} \cdot \vec{\theta}} \text{Tr} \left(e^{-t \Delta_{p,\theta}} \right) \frac{d^l \theta}{(2\pi)^l}. \quad (3.11)$$

From [32, Chapter XII], the eigenvalues $\lambda_i(\theta)$ form a sequence of nonnegative algebraic functions locally on $\widehat{\Gamma}$. (This corrects a claim in [15] that they are local analytic functions, which is only guaranteed if $l = 1$.) It follows that there is convergence in $L^1(\widehat{\Gamma})$:

$$\lim_{t \rightarrow \infty} \text{Tr} \left(e^{-t \Delta_{p,\theta}} \right) = b_p^{(2)}(M), \quad (3.12)$$

where $b_p^{(2)}(M)$ is the number of such algebraic functions which equal the zero function. Hence if $\vec{m} \neq 0$ then $b_{p,\langle \vec{m} \rangle}(M) = 0$. \square

4. PROOFS OF PROPOSITIONS 9 AND 10

Let Λ be the reduced group C^* -algebra of Γ . We assume that there is an algebra \mathfrak{B} such that

1. $\mathbb{C}\Gamma \subseteq \mathfrak{B} \subseteq \Lambda$.
2. \mathfrak{B} is the projective limit of a sequence

$$\dots \rightarrow B_{j+1} \rightarrow B_j \rightarrow \dots \rightarrow B_0 \quad (4.1)$$

of Banach algebras $(B_j, |\cdot|_j)$ with unit, and $B_0 = \Lambda$.

3. If $i_j : \mathfrak{B} \rightarrow B_j$ is the induced homomorphism then i_0 is injective with dense image and \mathfrak{B} is closed under the holomorphic functional calculus in Λ .

4. Given a conjugacy class $\langle g \rangle$ of Γ , define $\tau_{\langle g \rangle} : \mathbb{C}\Gamma \rightarrow \mathbb{C}$ as in (2.48). Then for all $\langle g \rangle \in \mathcal{C}$, $\tau_{\langle g \rangle}$ extends to a continuous linear functional on \mathfrak{B} , which we again denote by $\tau_{\langle g \rangle}$.

The topology on \mathfrak{B} comes from the submultiplicative seminorms $\|\cdot\|_j = |i_j(\cdot)|_j$. Condition 4 is equivalent to saying that $\text{HC}^0(\mathbb{C}\Gamma) = \text{HC}^0(\mathfrak{B})$. If Γ is a virtually nilpotent or Gromov-hyperbolic group then conditions 1-4 are known to be satisfied by the rapid-decay algebra \mathfrak{B} [12, p. 397].

If \mathfrak{E} is a finitely-generated right projective \mathfrak{B} -module then there is a continuous trace

$$\mathrm{Tr} : \mathrm{End}_{\mathfrak{B}}(\mathfrak{E}) \rightarrow \mathfrak{B}/[\overline{\mathfrak{B}}, \mathfrak{B}]. \quad (4.2)$$

Explicitly, suppose that $\mathfrak{E} = \mathfrak{B}^N e$ for some $N > 0$ and some projection $e \in M_N(\mathfrak{B})$. If $A \in \mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$, we can think of A as an element of $M_N(\mathfrak{B})$ satisfying $A = eA = Ae$. Then

$$\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii} \mod [\overline{\mathfrak{B}}, \mathfrak{B}]. \quad (4.3)$$

(We quotient by the closure of $[\mathfrak{B}, \mathfrak{B}]$ to ensure that the trace takes value in a Fréchet space.)

As Λ is a C^* -algebra, there is a calculus of Λ -pseudodifferential operators on M [23]. Suppose that E^1 is a smooth Λ -vector bundle on M , meaning the fibers of E^1 are all isomorphic to a fixed finitely-generated projective right Λ -module \mathfrak{E}^1 and the transition functions are smooth functions with value in $\mathrm{Aut}_{\Lambda}(\mathfrak{E}^1)$. Let E^2 be another smooth Λ -vector bundle on M . The elements of the pseudodifferential algebra $\Psi_{\Lambda}^{\infty}(M; E^1, E^2)$ map smooth sections of E^1 to smooth sections of E^2 and commute with the Λ -action.

In [17, Section 6.1] we extended this to a calculus of \mathfrak{B} -pseudodifferential operators and proved some basic properties of such operators. We only state the necessary facts, referring to [17] for details.

Let \mathcal{E}^1 and \mathcal{E}^2 be smooth \mathfrak{B} -vector bundles on M . By an extension of the Serre-Swan theorem, we can write $\mathcal{E}^1 = (M \times \mathfrak{B}^N)e^1$ for some $N > 0$ and some projection $e^1 \in C^{\infty}(M; M_N(\mathfrak{B}))$. Define a B_j -vector bundle by $E_j^1 = (M \times B_j^N)i_j(e^1)$. Then \mathcal{E}^1 is the projective limit of

$$\dots \rightarrow E_{j+1}^1 \rightarrow E_j^1 \rightarrow \dots \rightarrow E_0^1, \quad (4.4)$$

and similarly for \mathcal{E}^2 .

For each $j \geq 0$, there is an algebra $\Psi_{B_j}^{\infty}(M; E_j^1, E_j^2)$ of B_j -pseudodifferential operators. The algebra $\Psi_{\mathfrak{B}}^{\infty}(M; \mathcal{E}^1, \mathcal{E}^2)$ of \mathfrak{B} -pseudodifferential operators is the projective limit of

$$\dots \rightarrow \Psi_{B_{j+1}}^{\infty}(M; E_{j+1}^1, E_{j+1}^2) \rightarrow \Psi_{B_j}^{\infty}(M; E_j^1, E_j^2) \rightarrow \dots \rightarrow \Psi_{B_0}^{\infty}(M; E_0^1, E_0^2). \quad (4.5)$$

Let \mathcal{E} be a \mathfrak{B} -vector bundle on M . Given $T \in \Psi_{\mathfrak{B}}^{\infty}(M; \mathcal{E}, \mathcal{E})$, let $i_j(T)$ be its image in $\Psi_{B_j}^{\infty}(M; E_j, E_j)$.

Proposition 14. [17, Proposition 19] *If $i_0(T)$ is invertible in $\Psi_{B_0}^{\infty}(M; E_0, E_0)$ then T is invertible in $\Psi_{\mathfrak{B}}^{\infty}(M; \mathcal{E}, \mathcal{E})$.*

Note that $\Psi_{\mathfrak{B}}^{-\infty}(M; \mathcal{E}, \mathcal{E})$ is an algebra in its own right (without unit) of smoothing operators. Given $T \in \Psi_{\mathfrak{B}}^{-\infty}(M; \mathcal{E}, \mathcal{E})$, let $\sigma_{\Psi^{-\infty}}(T)$ denote its spectrum in $\Psi_{\mathfrak{B}}^{-\infty}(M; \mathcal{E}, \mathcal{E})$ and let $\sigma_{\Psi^{\infty}}(T)$ denote its spectrum in $\Psi_{\mathfrak{B}}^{\infty}(M; \mathcal{E}, \mathcal{E})$.

Lemma 3. [17, Lemma 2] $\sigma_{\Psi^{-\infty}}(T) = \sigma_{\Psi^{\infty}}(T)$.

Consider the algebra \mathfrak{A} of integral operators whose kernels $K(m_1, m_2) \in \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}_{m_2}, \mathcal{E}_{m_1})$ are continuous in m_1 and m_2 , with multiplication

$$(KK')(m_1, m_2) = \int_Z K(m_1, m)K'(m, m_2) d\mathrm{vol}(m). \quad (4.6)$$

Let A_j be the analogous algebra with continuous kernels $K(m_1, m_2) \in \text{Hom}_{B_j}((E_j)_{m_2}, (E_j)_{m_1})$. Give $\text{Hom}_{B_j}((E_j)_{m_2}, (E_j)_{m_1})$ the Banach space norm $|\cdot|_j$ induced from $\text{Hom}(B_j^N, B_j^N)$. Define a norm $|\cdot|_j$ on A_j by

$$|K|_j = (\text{vol}(M))^{-1} \max_{m_1, m_2 \in M} |K(m_1, m_2)|_j. \quad (4.7)$$

Then one can check that A_j is a Banach algebra (without unit). Furthermore, \mathfrak{A} is the projective limit of $\{A_j\}_{j \geq 0}$. Any smoothing operator $T \in \Psi_{\mathfrak{B}}^{-\infty}(M; \mathcal{E}, \mathcal{E})$ gives an element of \mathfrak{A} through its Schwartz kernel. Let $\sigma_{\mathfrak{A}}(T)$ be its spectrum in \mathfrak{A} .

Lemma 4. [17, Lemma 3] $\sigma_{\mathfrak{A}}(T) = \sigma_{\Psi^{-\infty}}(T)$.

Define a continuous trace $\text{TR} : \mathcal{A} \rightarrow \mathfrak{B}/[\overline{\mathfrak{B}}, \mathfrak{B}]$ by

$$\text{TR}(K) = \int_M \text{Tr}(K(m, m)) d\text{vol}(m). \quad (4.8)$$

Suppose that $0 \notin \text{spec}(\widehat{\Delta}_p)$ or that 0 is isolated in $\text{spec}(\widehat{\Delta}_p)$. Let \mathcal{D} be the \mathfrak{B} -vector bundle $\widehat{M} \times_{\Gamma} \mathfrak{B}$ on M and put $\mathcal{E} = \Lambda^p(T^*M) \otimes \mathcal{D}$. We can lift Δ_p from M to a differential operator $\widetilde{\Delta}_p \in \Psi_{\mathfrak{B}}^2(M; \mathcal{E}, \mathcal{E})$. Then for all $t > 0$, $e^{-t\widetilde{\Delta}_p} \in \Psi_{\mathfrak{B}}^{-\infty}(M; \mathcal{E}, \mathcal{E})$. As in [13, Section 3], one can show that

$$\text{Tr}_{\langle g \rangle} \left(e^{-t\widehat{\Delta}_p} \right) = \tau_{\langle g \rangle} \left(\text{TR} \left(e^{-t\widetilde{\Delta}_p} \right) \right). \quad (4.9)$$

Let E be the analogous Λ -vector bundle on M whose fiber over $m \in M$ is isomorphic to $\Lambda^p(T_m^*M) \otimes \Lambda$. Recall that $i_0(\widetilde{\Delta}_p)$ is the extension of $\widetilde{\Delta}_p$ to an element of $\Psi_{\Lambda}^2(M; E, E)$. Let $N(\Gamma)$ denote the group von Neumann algebra of Γ [1]. Let \overline{E} be the natural $N(\Gamma)$ -vector bundle on M whose fiber over $m \in M$ is isomorphic to $\Lambda^p(T_m^*M) \otimes N(\Gamma)$. Let $\overline{\Delta}_p$ be the extension of $\widetilde{\Delta}_p$ to an element of $\Psi_{N(\Gamma)}^2(M; \overline{E}, \overline{E})$. As $L^2(\widehat{M})$ is isomorphic to the L^2 -sections of the Hilbert bundle $\widehat{M} \times_{\Gamma} l^2(\Gamma)$, it follows that $\sigma(\widehat{\Delta}_p) = \sigma(\overline{\Delta}_p)$. As the C^* -algebra Λ is a closed subalgebra of $N(\Gamma)$, it follows that $\sigma(\overline{\Delta}_p) = \sigma(i_0(\widetilde{\Delta}_p))$. Using Proposition 14, it now follows that $0 \notin \sigma(\widetilde{\Delta}_p)$ or that 0 is isolated in $\sigma(\widetilde{\Delta}_p)$. Let c be a small loop around $0 \in \mathbb{C}$, oriented counterclockwise. The projection onto $\text{Ker}(\widetilde{\Delta}_p)$ is

$$\Pi_{\text{Ker}(\widetilde{\Delta}_p)} = \frac{1}{2\pi i} \int_c \frac{dz}{z - \widetilde{\Delta}_p}. \quad (4.10)$$

It follows from arguments as in [23] that $\text{Ker}(\widetilde{\Delta}_p)$ is a finitely-generated projective right \mathfrak{B} -module. Let $\widetilde{\Delta}'_p$ be the compression of $\widetilde{\Delta}_p$ onto $\text{Im} \left(I - \Pi_{\text{Ker}(\widetilde{\Delta}_p)} \right)$. Then in terms of the trace of (4.2),

$$\text{Tr}_{\langle g \rangle} \left(e^{-t\widehat{\Delta}_p} \right) = \tau_{\langle g \rangle} \left(\text{Tr} \left(I_{\text{Ker}(\widetilde{\Delta}_p)} \right) \right) + \tau_{\langle g \rangle} \left(\text{TR} \left(e^{-t\widetilde{\Delta}'_p} \right) \right). \quad (4.11)$$

As τ and TR are continuous, it suffices to show that there is some j such that the A_j -norm of $e^{-t\widetilde{\Delta}'_p}$ is rapidly-decreasing in t .

As $\{e^{-t\tilde{\Delta}'_p}\}_{t>0}$ gives a 1-parameter semigroup in the Banach algebra A_j , it follows from [7, Theorem 1.22] that the number

$$a = \lim_{t \rightarrow \infty} t^{-1} \ln \left| e^{-t\tilde{\Delta}'_p} \right|_j \quad (4.12)$$

exists. Furthermore, for all $t > 0$, the spectral radius of $e^{-t\tilde{\Delta}'_p}$ is e^{at} . Let $\lambda_0 > 0$ be the infimum of the spectrum of the generator $\tilde{\Delta}'_p$. Then by the spectral mapping theorem, the spectral radius of $e^{-t\tilde{\Delta}'_p}$ is $e^{-t\lambda_0}$. Thus there is a constant $C > 0$ such that for $t > 1$,

$$\left| e^{-t\tilde{\Delta}'_p} \right|_j \leq C e^{-\lambda_0 t/2}. \quad (4.13)$$

Hence

$$b_{p,\langle g \rangle}(M) = \tau_{\langle g \rangle} \left(\text{Tr} \left(I_{\text{Ker}(\tilde{\Delta}_p)} \right) \right) \quad (4.14)$$

is well-defined. By similar arguments one can justify (2.13), showing that $b_{p,\langle g \rangle}(M)$ is metric-independent.

Now let D be a Dirac-type operator on M such that $0 \notin \text{spec}(\hat{D})$ or 0 is isolated in $\text{spec}(\hat{D})$. Put $\mathcal{E} = S \otimes V \otimes \mathcal{D}$. We can lift D to a differential operator $\tilde{D} \in \Psi^1_{\mathfrak{B}}(M; \mathcal{E}, \mathcal{E})$. Then for all $s > 0$, $\tilde{D}e^{-s^2\tilde{D}^2} \in \Psi^{-\infty}_{\mathfrak{B}}(M; \mathcal{E}, \mathcal{E})$. As in [13, Section 3], one can show that

$$\eta_{\langle g \rangle}(s) = \tau_{\langle g \rangle} \left(\text{TR} \left(\tilde{D}e^{-s^2\tilde{D}^2} \right) \right). \quad (4.15)$$

From finite-propagation estimates, we know that $\eta_{\langle g \rangle}(s)$ is integrable for small- s . Hence we must show that $\tau_{\langle g \rangle} \left(\text{TR} \left(\tilde{D}e^{-s^2\tilde{D}^2} \right) \right)$ is integrable for large- s . It suffices to show that there is some j such that the A_j -norm of $\tilde{D}e^{-s^2\tilde{D}^2}$ is rapidly-decreasing in s .

Let E be the natural Λ -vector bundle on M whose fiber over $m \in M$ is isomorphic to $S_m \otimes V_m \otimes \Lambda$. Recall that $i_0(\tilde{D})$ is the extension of \tilde{D} to an element of $\Psi^1_{\Lambda}(M; E, E)$. As before, $\sigma(\hat{D}) = \sigma(i_0(\tilde{D}))$. Using Proposition 14, it now follows that $0 \notin \sigma(\tilde{D})$ or that 0 is isolated in $\sigma(\tilde{D})$. Let c be a small loop around $0 \in \mathbb{C}$, oriented counterclockwise. The projection on $\text{Ker}(\tilde{D})$ is

$$\Pi_{\text{Ker}(\tilde{D})} = \frac{1}{2\pi i} \int_c \frac{dz}{z - \tilde{D}}. \quad (4.16)$$

Let \tilde{D}' be the compression of \tilde{D} onto $\text{Im} \left(I - \Pi_{\text{Ker}(\tilde{D})} \right)$. Then

$$\tilde{D}e^{-s^2\tilde{D}^2} = 0_{\text{Ker}(\tilde{D})} \oplus \tilde{D}'e^{-s^2\tilde{D}'^2}. \quad (4.17)$$

Hence we may as well assume that $0 \notin \text{spec}(\tilde{D})$. Let $\lambda_0 > 0$ be the infimum of the spectrum of the generator \tilde{D}^2 . As in (4.13), there is a constant $C > 0$ such that for $t > 1$,

$$\left| e^{-t\tilde{D}^2} \right|_j \leq C e^{-\lambda_0 t/2}. \quad (4.18)$$

As

$$\left| \tilde{D}e^{-s^2\tilde{D}^2} \right|_j \leq \left| \tilde{D}e^{-\tilde{D}^2} \right|_j \cdot \left| e^{-(s^2-1)\tilde{D}^2} \right|_j, \quad (4.19)$$

it follows that $\eta_{\langle g \rangle}(s)$ is large- s integrable.

Suppose that $\{ds^2(u)\}_{u \in [-1,1]}$ is a smooth 1-parameter of positive-scalar-curvature metrics on M . Let $D(u)$ be the Dirac operator on M . Then for all $u \in [-1,1]$, $\widehat{D}(u)$ is invertible. Using the above methods, one sees that

$$\lim_{s \rightarrow \infty} s \tau_{\langle g \rangle} \left(\text{TR} \left(\frac{d\widetilde{D}}{du} e^{-s^2 \widetilde{D}^2} \right) \right) = 0. \quad (4.20)$$

From Proposition 3, $\eta_{\langle g \rangle}(M)$ is independent of u . \square

Remark : If Γ is virtually nilpotent, one can also prove Propositions 9 and 10 using finite propagation speed estimates on \widehat{M} , as in [14]. This does not work when Γ is Gromov-hyperbolic, which is why we use the more indirect method of proof above.

5. PROOF OF PROPOSITION 11

As in [8, Section 2], the Selberg trace formula implies that there are functions $\{G_j(t)\}_{j=0}^{d-1}$ of the form

$$G_j(t) = a_j t^{-1/2} e^{-\frac{l^2}{4t}} e^{-tc_j^2} \quad (5.1)$$

so that for $0 \leq j \leq d$,

$$\text{Tr}_{\langle g \rangle} \left(e^{-t \widehat{\Delta}_j} \right) = G_j(t) + G_{j-1}(t). \quad (5.2)$$

Here a_j and c_j are nonnegative constants whose exact values are not important for the moment. It is clear from (5.1) and (5.2) that $b_{p,\langle g \rangle}(M)$ vanishes for all p .

Now suppose that the dimension of M is $d = 2n + 1$. The Selberg trace formula gives the following result.

Proposition 15. [8, Theorem 2] *For $0 \leq j \leq 2n$, put $c_j = |n - j|$ and*

$$G_t(\sigma_j) = \frac{\text{Tr}(\sigma_j(m))}{k \det(I - e^{-l}m)} \frac{1}{\sqrt{4\pi t}} e^{-\frac{l^2}{4t}} e^{-tc_j^2} e^{-nl}. \quad (5.3)$$

Then for $0 \leq j \leq d$,

$$\text{Tr}_{\langle g \rangle} \left(e^{-t \widehat{\Delta}_j} \right) = G_t(\sigma_j) + G_t(\sigma_{j-1}). \quad (5.4)$$

Hence from (2.14),

$$\begin{aligned} \mathcal{T}_{\langle g \rangle}(t) &= \sum_{p=0}^d (-1)^j j [G_t(\sigma_j) + G_t(\sigma_{j-1})] \\ &= \sum_{j=0}^{2n} (-1)^{j+1} G_t(\sigma_j). \end{aligned} \quad (5.5)$$

For $l > 0$,

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{l^2}{4t}} e^{-tc^2} \frac{dt}{t} = \frac{e^{-lc}}{l}. \quad (5.6)$$

Equation (2.34) now follows from (2.16).

For $r > 0$, we have

$$\begin{aligned} \mathcal{T}_{\langle g^r \rangle}(M) &= \frac{e^{-nrl}}{k \det(I - e^{-rl} m^r)} \sum_{j=0}^{2n} (-1)^j e^{-rl|n-j|} \text{Tr}(\sigma_j(m^r)) \\ &= \frac{(-1)^n}{k} e^{-nrl} \text{Tr}(\sigma_n(m^r)) + O(e^{-2nrl}). \end{aligned} \quad (5.7)$$

Then

$$l = \frac{1}{n} \sup\{\alpha \in \mathbb{R} : |\mathcal{T}_{\langle g^r \rangle}(M)| = O(e^{-\alpha r})\} \quad (5.8)$$

Hence one recovers the marked length spectrum of M from $\{\mathcal{T}_{\langle g \rangle}(M)\}_{\langle g \rangle \in \mathcal{C}}$.

Let D be the tangential signature operator. By [21, Theorem 2.1],

$$\eta_{\langle g \rangle}(s) = (2i)^n \frac{2\pi i}{k} \frac{l^2 \sin(\theta_1) \dots \sin(\theta_n)}{|\mu_1 - \mu_1^{-1}|^2 \dots |\mu_n - \mu_n^{-1}|^2} \frac{e^{-\frac{l^2}{4s^2}}}{(4\pi)^{3/2} s^3}. \quad (5.9)$$

Equation (2.36) now follows from (2.22).

Equation (2.37) comes from a straightforward computation. \square

Remark : Using the results of Moscovici and Stanton [24, 25], one can extend Proposition 11 to general nonpositively-curved locally symmetric spaces.

6. PROOF OF PROPOSITION 12

Let $\pi : M \rightarrow S^1$ be the natural projection map. For $e^{i\theta} \in U(1)$, let E_θ be the flat complex line bundle on S^1 with holonomy $e^{i\theta}$. Let $T(\theta) \in \mathbb{R}$ be the Ray-Singer analytic torsion of M , computed with the flat bundle $\pi^*(E_\theta)$. As in [15, Section VI], it follows from Fourier analysis that

$$\mathcal{T}_{\langle k \rangle}(M) = \int_{S^1} e^{-ik\theta} T(\theta) \frac{d\theta}{2\pi}. \quad (6.1)$$

From [22, Section 3] and the Cheeger-Müller theorem [4, 27],

$$T(\theta) = \sum_{p=0}^n (-1)^p \ln |\det(I - e^{i\theta} \phi_p^*)|^{-2}. \quad (6.2)$$

Given $\lambda \neq 0$, if $k > 0$ then

$$\int_{S^1} e^{-ik\theta} \ln |1 - e^{i\theta} \lambda|^{-2} \frac{d\theta}{2\pi} = \frac{f(\lambda^k)}{k} \quad (6.3)$$

and if $k < 0$ then

$$\int_{S^1} e^{-ik\theta} \ln |1 - e^{i\theta} \lambda|^{-2} \frac{d\theta}{2\pi} = -\frac{f(\bar{\lambda}^{-k})}{k}. \quad (6.4)$$

Equation (2.40) follows from combining (6.1)-(6.4).

By standard arguments,

$$\zeta(z) = \prod_{p=0}^n \det(I - z\phi_p^*)^{(-1)^{p+1}}. \quad (6.5)$$

Equation (2.41) follows from (6.1), (6.2) and (6.5).

Now suppose that ϕ preserves D_Z . It follows that ϕ is an isometry of Z with respect to the Riemannian metric defining D_Z . In terms of the coordinates (u, z) on $\widehat{M} = \mathbb{R} \times Z$, we can write

$$\widehat{D} = \begin{pmatrix} -i\partial_u & D_{Z,-} \\ D_{Z,+} & i\partial_u \end{pmatrix}. \quad (6.6)$$

Then

$$\widehat{D}^2 = \begin{pmatrix} -\partial_u^2 + D_{Z,-}D_{Z,+} & 0 \\ 0 & -\partial_u^2 + D_{Z,+}D_{Z,-} \end{pmatrix} \quad (6.7)$$

and

$$e^{-s^2\widehat{D}^2}((u, z), (u', z')) = \begin{pmatrix} \frac{1}{\sqrt{4\pi s^2}} e^{-\frac{(u-u')^2}{4s^2}} e^{-s^2 D_{Z,-}D_{Z,+}}(z, z') & 0 \\ 0 & \frac{1}{\sqrt{4\pi s^2}} e^{-\frac{(u-u')^2}{4s^2}} e^{-s^2 D_{Z,+}D_{Z,-}}(z, z') \end{pmatrix}. \quad (6.8)$$

It follows that

$$\mathrm{tr} \left(\widehat{D} e^{-s^2\widehat{D}^2}((u, z), (u', z')) \right) = i \frac{1}{\sqrt{4\pi s^2}} \frac{u - u'}{2s^2} e^{-\frac{(u-u')^2}{4s^2}} \mathrm{tr}_s \left(e^{-s^2 D_Z^2}(z, z') \right). \quad (6.9)$$

Hence

$$\begin{aligned} \eta_{\langle k \rangle}(M) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_Z i \frac{1}{\sqrt{4\pi s^2}} \frac{k}{2s^2} e^{-\frac{k^2}{4s^2}} \mathrm{tr}_s \left(e^{-s^2 D_Z^2}(\phi^k(z), z) \right) d\mathrm{vol}(z) ds \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty i \frac{1}{\sqrt{4\pi s^2}} \frac{k}{2s^2} e^{-\frac{k^2}{4s^2}} \mathrm{Tr}_s \left(\phi^k e^{-s^2 D_Z^2} \right) ds \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty i \frac{1}{\sqrt{4\pi s^2}} \frac{k}{2s^2} e^{-\frac{k^2}{4s^2}} \mathrm{Tr}_s \left(\phi^k \Big|_{\mathrm{Ker}(D_Z)} \right) ds \\ &= \frac{i}{k\pi} \mathrm{Tr}_s \left(\phi^k \Big|_{\mathrm{Ker}(D_Z)} \right) \int_0^\infty e^{-\frac{k^2}{4s^2}} d \left(-\frac{k^2}{4s^2} \right) \\ &= \frac{i}{k\pi} \mathrm{Tr}_s \left(\phi^k \Big|_{\mathrm{Ker}(D_Z)} \right). \end{aligned} \quad (6.10)$$

The proposition follows. \square

7. PROOF OF PROPOSITION 13

Any irreducible unitary representation ρ of $F \widetilde{\times}_\alpha \mathbb{Z}$ arises as follows [19, Section 10]. First, \mathbb{Z} acts on the dual space \widehat{F} . A periodic point of period j corresponds to a representation $\mu : F \rightarrow U(N)$ and a matrix $U \in U(N)$ such that $\mu(\alpha^j(f)) = U\mu(f)U^{-1}$. (The matrix U is determined up to multiplication by a unit complex number.) Consider the representation $\nu : F \widetilde{\times}_\alpha j\mathbb{Z} \rightarrow U(N)$ given by

$$\nu(f, jr) = \mu(f) U^r. \quad (7.1)$$

Then ρ comes from inducing ν from $F \widetilde{\times}_\alpha j\mathbb{Z}$ to $F \widetilde{\times}_\alpha \mathbb{Z}$. The character of ρ is

$$\chi_\rho(f, k) = \begin{cases} 0 & \text{if } j \nmid k, \\ \text{Tr}([\mu(f) + \mu(\alpha^{-1}(f)) + \dots + \mu(\alpha^{-(j-1)}(f))] U^r) & \text{if } k = jr. \end{cases} \quad (7.2)$$

Let $\mathcal{T}_\rho(M)$ be the analytic torsion of M , computed using the representation ρ . From Fourier analysis,

$$\mathcal{T}_\rho(M) = \sum_{f, k} \chi_\rho(f, k) \mathcal{T}_{(f, k)}(M). \quad (7.3)$$

Let M' be the mapping torus of ϕ^j . Then M' is a j -fold cover of M . By [9, (VI), p. 27],

$$\mathcal{T}_\rho(M) = \mathcal{T}_\nu(M'). \quad (7.4)$$

Let E_μ be the flat \mathbb{C}^N -bundle on Z coming from the representation μ . Then ϕ^j acts on Z and preserves E_μ . Let $L_\mu(r)$ be the Lefschetz number of ϕ^{jr} acting on (Z, E_μ) . Put

$$\zeta_\nu(z) = \exp \left(\sum_{r>0} \frac{z^r}{r} L_\mu(r) \right). \quad (7.5)$$

It follows from [22, Section 3] that

$$\mathcal{T}_\nu(M') = \ln |\zeta_\nu(1)|^2. \quad (7.6)$$

Take a cellular decomposition of Z . Let \widehat{Z} have the lifted cellular structure and let $C^*(\widehat{Z})$ denote the cellular cochains on \widehat{Z} . We let F act on $C^*(\widehat{Z})$ on the right by

$$\omega \cdot f = R_{f^{-1}}^* \omega. \quad (7.7)$$

Then

$$\widehat{\phi}^*(\omega \cdot f) = (\widehat{\phi}^* \omega) \cdot \alpha(f). \quad (7.8)$$

We can identify $C^*(Z; E_\mu)$ with $C^*(\widehat{Z}) \otimes_F \mathbb{C}^N$, with the relation $\omega \cdot f \otimes_F v = \omega \otimes_F \mu(f)v$. Then ϕ^{jr} acts on $C^*(\widehat{Z}) \otimes_F \mathbb{C}^N$ by

$$\phi^{jr}(\omega \otimes_F v) = (\widehat{\phi}^{jr})^* \omega \otimes_F U^r v. \quad (7.9)$$

Letting Tr_s denote the supertrace on $C^*(Z; E_\mu)$, we want to compute

$$L_\mu(r) = \text{Tr}_s(\phi^{jr}). \quad (7.10)$$

For the moment we concentrate on $C^p(Z; E_\mu)$. Let $\{e_i\}$ be a basis of $C^p(\widehat{Z})$ consisting of dual p -cells. The set of such dual p -cells has a free F -action. Write the action of $\widehat{\phi}^*$ on $C^p(\widehat{Z})$ as

$$\widehat{\phi}^*(e_i) = \sum_l \widehat{\phi}_{e_i \rightarrow e_l}^* e_l. \quad (7.11)$$

From (7.8),

$$\widehat{\phi}_{e_i \rightarrow e_l}^* = \widehat{\phi}_{e_i f \rightarrow e_l \alpha(f)}^*. \quad (7.12)$$

Let $\{\bar{e}_i\}$ be a set of representatives for the F -orbits of the dual p -cells. Then as a vector space,

$$C^p(\widehat{Z}) \otimes_F \mathbb{C}^N = \bigoplus_i \bar{e}_i \otimes \mathbb{C}^N \quad (7.13)$$

and so

$$\begin{aligned} \phi^{jr}(\bar{e}_i \otimes v) &= \sum_{l,f} \left(\widehat{\phi}^{jr} \right)_{\bar{e}_i \rightarrow \bar{e}_l f}^* \bar{e}_l f \otimes_F U^r v \\ &= \sum_{l,f} \left(\widehat{\phi}^{jr} \right)_{\bar{e}_i \rightarrow \bar{e}_l f}^* \bar{e}_l \otimes \mu(f) U^r v. \end{aligned} \quad (7.14)$$

Hence

$$\mathrm{Tr}(\phi^{jr}) = \sum_{i,f} \left(\widehat{\phi}^{jr} \right)_{\bar{e}_i \rightarrow \bar{e}_i f}^* \mathrm{Tr}(\mu(f) U^r). \quad (7.15)$$

As the choice of the representatives $\{\bar{e}_i\}$ is arbitrary, we can also write

$$\mathrm{Tr}(\phi^{jr}) = \frac{1}{|F|} \sum_{i,f} \left(\widehat{\phi}^{jr} \right)_{e_i \rightarrow e_i f}^* \mathrm{Tr}(\mu(f) U^r). \quad (7.16)$$

We have

$$\begin{aligned} \mathrm{Tr}(\phi^{jr}) &= \frac{1}{|F|} \sum_{i,f} \left(\widehat{\phi}^{jr} \right)_{e_i \rightarrow e_i f}^* \mathrm{Tr}(\mu(f) U^r) \\ &= \frac{1}{|F|} \sum_{i,l,f} \left(\widehat{\phi} \right)_{e_i \rightarrow e_l}^* \left(\widehat{\phi}^{jr-1} \right)_{e_l \rightarrow e_i f}^* \mathrm{Tr}(\mu(f) U^r) \\ &= \frac{1}{|F|} \sum_{i,l,f} \left(\widehat{\phi}^{jr-1} \right)_{e_l \rightarrow e_i f}^* \left(\widehat{\phi} \right)_{e_i \rightarrow e_l}^* \mathrm{Tr}(\mu(f) U^r) \\ &= \frac{1}{|F|} \sum_{i,l,f} \left(\widehat{\phi}^{jr-1} \right)_{e_l \rightarrow e_i f}^* \left(\widehat{\phi} \right)_{e_i f \rightarrow e_l \alpha(f)}^* \mathrm{Tr}(\mu(f) U^r) \\ &= \frac{1}{|F|} \sum_{l,f} \left(\widehat{\phi}^{jr} \right)_{e_l \rightarrow e_l \alpha(f)}^* \mathrm{Tr}(\mu(f) U^r) \\ &= \frac{1}{|F|} \sum_{l,f} \left(\widehat{\phi}^{jr} \right)_{e_l \rightarrow e_l f}^* \mathrm{Tr}(\mu(\alpha^{-1}(f)) U^r). \end{aligned} \quad (7.17)$$

Then from (7.16) and (7.17),

$$\mathrm{Tr}(\phi^{jr}) = \frac{1}{j|F|} \sum_{i,f} \left(\widehat{\phi}^{jr} \right)_{e_i \rightarrow e_i f}^* \mathrm{Tr}([\mu(f) + \mu(\alpha^{-1}(f)) + \dots + \mu(\alpha^{-(j-1)}(f))] U^r). \quad (7.18)$$

Put

$$n_{p,jr}(f) = \frac{1}{|F|} \sum_{i,f} \left(\widehat{\phi}^{jr} \right)_{e_i \rightarrow e_i f}^*. \quad (7.19)$$

From (7.2), (7.10) and (7.18),

$$L_\mu(r) = \frac{1}{j} \sum_f \chi_\rho(f, jr) \sum_{p=0}^n (-1)^p n_{p,jr}(f). \quad (7.20)$$

Put

$$i_{p,jr}(f) = \sum_{f' \sim_{jr} f} \sum_i \left(\widehat{\phi}^{jr} \right)_{\bar{e}_i \rightarrow \bar{e}_i f}^*. \quad (7.21)$$

The Nielsen fixed-point index $I_{jr}(f) \in \mathbb{Z}$ of the transformation ϕ^{jr} is defined by [10, Section 1]

$$I_{jr}(f) = \sum_{p=0}^n (-1)^p i_{p,jr}(f). \quad (7.22)$$

Put $s_{jr}(f) = |\{\gamma \in F : \gamma f \alpha^{jr}(\gamma^{-1}) = f\}|$. We have

$$\begin{aligned} i_{p,jr}(f) &= \frac{1}{s_{jr}(f)} \sum_{i,\gamma} \left(\widehat{\phi}^{jr} \right)_{\bar{e}_i \rightarrow \bar{e}_i \gamma f \alpha^{jr}(\gamma^{-1})}^* \\ &= \frac{1}{s_{jr}(f)} \sum_{i,\gamma} \left(\widehat{\phi}^{jr} \right)_{\bar{e}_i \gamma \rightarrow \bar{e}_i \gamma f}^* \\ &= \frac{1}{s_{jr}(f)} \sum_i \left(\widehat{\phi}^{jr} \right)_{e_i \rightarrow e_i f}^*. \end{aligned} \quad (7.23)$$

Then from (7.19) and (7.23),

$$n_{p,jr}(f) = \frac{s_{jr}(f)}{|F|} i_{p,jr}(f) = \frac{i_{p,jr}(f)}{|[f]_{jr}|}. \quad (7.24)$$

Hence the Lefschetz number is given in terms of the Nielsen index by

$$L_\mu(r) = \frac{1}{j} \sum_f \frac{\chi_\rho(f, jr) I_{jr}(f)}{|[f]_{jr}|}. \quad (7.25)$$

Substituting (7.25) into (7.6) and using (7.3)-(7.4) gives (2.46). As Γ is a type-I discrete group [19, p. 61], knowing equation (2.46) for all $\rho \in \widehat{\Gamma}$ determines $\{\mathcal{T}_{(f,k)}(M)\}_{(f,k) \in \Gamma}$. \square

8. EXAMPLES

Proposition 6 : It follows from Proposition 4.1 that $b_{p,\langle g \rangle}(M)$ can be nonzero if Γ is finite. For example, $b_{0,\langle g \rangle}(M) = \frac{|\langle g \rangle|}{|\Gamma|}$.

Proposition 7 : It follows from Proposition 4.2 that $\mathcal{T}_{\langle g \rangle}(M)$ is nonzero in some examples in which M is a lens space.

Proposition 8 : It follows from Proposition 4.3 that $\eta_{\langle g \rangle}(M)$ is nonzero in some examples in which M is a lens space, both for the tangential signature operator and the Dirac operator.

Proposition 9 : An even-dimensional closed hyperbolic manifold M satisfies the hypotheses

of the proposition for all p .

Proposition 10 : Let N_1 be a closed even-dimensional spin manifold whose fundamental group is virtually nilpotent or Gromov-hyperbolic, with $\hat{A}(N_1) \neq 0$. Let N_2 be a lens space which is spin and whose Dirac operator has a nonzero ρ -invariant. Put $M = N_1 \times N_2$. Shrink N_2 so that M has positive scalar curvature. Then M satisfies the hypotheses of the proposition and $\eta_{(e,g)}(M) = \hat{A}(N_1) \eta_{(g)}(N_2)$ is nonzero for appropriate g .

Proposition 11 : There are many nontrivial examples.

Proposition 12 : Nontrivial examples come from closed even-dimensional oriented manifolds Z with a finite-order orientation-preserving diffeomorphism ϕ such that ϕ has nonzero Lefschetz or Atiyah-Bott numbers.

Proposition 13 : Any example of Proposition 12 gives an example of Proposition 13 by taking F to be the trivial group. There are also many examples with F nontrivial.

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